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An Invariance and Closed Form Analysis of the Nonlinear Biharmonic Beam Equation

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Abstract

In this paper, we study the one-parameter Lie groups of point transformations that leave invariant the biharmonic partial differential equation (PDE) $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f(u)$. To this end, we construct the Lie and Noether symmetry generators and present reductions of biharmonic PDE. When f is arbitrary function of u, we obtain the solution of biharmonic equation in terms of Green function. The equation is further analysed when f is exponential function and for general power law. Furthermore, we use Noether's theorem and the 'multiplier approach' to construct conservation laws of the PDE.

Keywords: biharmonic equation; transformation groups; Lie symmetries; conservation laws.

1 Introduction

The biharmonic equation is a fourth-order partial differential equation that has variety of applications in applied mathematics. It is studied in the theory of elasticity, mechanics of elastic plates and in the slow motion of viscous fluids, inter alia [11, 15]. Moreover, it is used in the modelling of thin structures which can react elastically to external forces. The term biharmonic indicates that the function satisfying this equation satisfy the Laplace equation twice explicitly. The earliest applications of biharmonic equation deal with the classical theory of elastic plates developed by Bernouli, Euler, Lagrange, Germain, Poisson, Navier, Cauchy and Lamé. Moreover, Kirchhoff, Levy, J. C. Maxwell and Sir Horace Lamb developed the mathematical modelling of the theory of plates. The reduction of the analysis for two dimensional problem to the solutions of the biharmonic equation is due to Airy, who used calculations in the design of the structural support system for an astronomical telescope. Biharmonic equation and fourth order differential equations are important topic for many researchers. For instance, Lie symmetries of homogeneous biharmonic linear equation were given by Bluman et al. [3], while Bokhari et al. [4] considered symmetries and integrability of the fourth order Euler Bernouli beam equation and Sripana and Chatanin [16] studied Lie symmetry analysis and exact solutions to the quintic nonlinear beam equation. Moreover, symmetry analysis of fourth order noise reduction partial differential equations were studied by Leach [10].

The nonlinear biharmonic equation in two independent variables x, y and one dependent variable u is given by

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f(u). \tag{1}$$

We will compute the Lie & Noether symmetries and obtain reductions of biharmonic PDEs to ODEs. Moreover, we find conservation laws by using celebrated Noether's theorem and also by means of multipliers approach. The article is arranged as, we firstly, present definitions and results that will be used in this sequel. In Section 2, the determining equations associated with biharmonic PDE are obtained, Lie symmetries are calculated and reductions of biharmonic PDEs to ODEs are presented. Noether symmetries and conservation laws are computed in Sections 3 and 4, respectively.

Recall that, the Lie point symmetry of a system of differential equation is a local group of transformations which maps each solution of the system to its another solution. These symmetries are used to solve the differential equations. Moreover, Lie symmetries are applied to reduce the order of differential equations and to obtain the conservation laws.

Variational symmetries or symmetries of Lagrangian also known as Noether symmetries and Lie symmetries or symmetries of corresponding Euler-Lagrange equations (generally symmetries of differential equations) are considerably studied in the literature ([1], [5]). Moreover, Lie symmetry method is powerful tool for solving differential equations, [6]. It is used to reduce the systems of differential equations into equivalent systems of simpler forms. Likewise, these methods are used to reduce the order of differential equations and reductions of number of independent variables in case of PDEs. The Noether symmetries are more powerful due to the fact that they can give double reductions of differential equations [9]. In addition conservation laws are either obtained from these symmetries by mean of celebrated Noether's theorem ([8], [17]) or by direct construction methods [2] or by partial Lagrangian approach [7].

2 Lie Symmetries

We suppose, the PDE in equation (1) admits the one-parameter Lie group of point transformations with infinitesimal generator (vector field) given by,

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \zeta(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u}. \tag{2}$$

The fourth prolongation of the generator X in equation (2) is given by,

$$X^{[4]} = X + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_y^{(1)}\frac{\partial}{\partial u_y} + \eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}} + \ldots + \eta_{xxyy}^{(4)}\frac{\partial}{\partial u_{xxyy}} + \eta_{yyyy}^{(4)}\frac{\partial}{\partial u_{yyyy}}, \tag{3}$$

here, the extended infinitesimals $\eta^{(k)}$ satisfies following relations,

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \tag{4}$$

$$\eta_{i_1 i_2 \ldots i_k}^{(k)} = D_{i_k}\eta_{i_1 i_2 \ldots i_{k-1}}^{(k-1)} - (D_{i_k}\xi_j)u_{i_1 i_2 \ldots i_{k-1} j}, \tag{5}$$

$k = 1, 2, 3, 4$ and D_i being total derivative operator [3].

Here, the invariance condition is given by

$$X^{[4]}[u_{xxxx} + 2u_{xxyy} + u_{yyyy} - f(u)]|_{u_{xxxx}+2u_{xxyy}+u_{yyyy}-f(u)=0}. \tag{6}$$

The above equation leads to symmetry determining equation given by

$$[\eta_{xxxx}^{(4)} + 2\eta_{xxyy}^{(4)} + \eta_{yyyy}^{(4)} - \eta f'(u)]|_{(1)} = 0. \tag{7}$$

Now we consider some cases for the function $f(u)$.

Case I: If f is arbitrary function of u then from symmetry determining equation (7) we get the following admitted Lie point symmetry generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{8}$$

which is three dimensional Lie Algebra. The commutation relations satisfied by these symmetry generators are given in Table 1.

Table 1: Lie brackets for the admitted Lie point symmetries of equation (1).

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	$-X_1$
X_3	$-X_2$	X_1	0

Reduction under X_1, X_2

Since $[X_1, X_2] = 0$, so either of X_1 or X_2 can be used to start reduction with. We begin with X_1. The characteristic equation associated with $X_1 = \frac{\partial}{\partial x}$ is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{du}{0}. \tag{9}$$

So we have $y = r$ and $u = w(r)$, therefore equation (1) reduces to

$$\frac{d^4 w}{dr^4} = f(w). \tag{10}$$

The symmetry analysis of this equation is given by Bokhari et al [4]. For the present case the equation (10) has symmetry generator

$$\frac{\partial}{\partial r}. \tag{11}$$

Moreover, in this case, the Lie reduction gives rise to the third order ODE given by

$$\beta\left(\frac{d}{d\alpha}\left(\beta\frac{d}{d\alpha}\left(\beta\frac{d\beta}{d\alpha}\right)\right)\right) = f(\alpha). \tag{12}$$

Here, $\alpha = w$ and $\beta = w'$ are invariants of translation group generated by $\frac{\partial}{\partial r}$. In the absence of further symmetries, one cannot further proceed. Reduction for the symmetry generator X_3 is

$$16s^2\frac{d^4 w}{ds^4} + 64s\frac{d^3 w}{ds^3} + 32\frac{d^2 w}{ds^2} = f(w). \tag{13}$$

By using $v = \frac{d^2 w}{ds^2}$, the above equation takes the form

$$16s^2\frac{d^2 v}{ds^2} + 64s\frac{dv}{ds} + 32v = h(v). \tag{14}$$

The solution of equation (14) subject to separated boundary conditions $u(a) = 0$ and $u(b) = 0$ is given by

$$v = \int_a^b G(s,t)h(v)dt. \tag{15}$$

Here $G(s,t)$ is the Green function associated with the equation (14) and is given by

$$G(s,t) = \begin{cases} \dfrac{(a-s)(b-t)}{x^2(b-a)}, & \text{if } a \le s < t, \\ \dfrac{(b-s)(a-t)}{x^2(b-a)}, & \text{if } t \le s \le b. \end{cases} \tag{16}$$

Case II: If f is an exponential function of the form $f = \delta e^u$, where $\delta = \pm 1$, then the equation (1) can be expressed as

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \delta e^u. \tag{17}$$

The admitted Lie point symmetries in this case are

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y}, \quad Y_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \quad Y_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{18}$$

which is four dimensional Lie algebra. The commutation relations for this case are expressed in following Table 2.

Reduction under Y_3, Y_4

Here $[Y_3, Y_4] = 0$, so we can have reduction with either Y_3 or Y_4. We start with Y_4. The Characteristic equation corresponding to $Y_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ is

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{du}{0}. \tag{19}$$

Table 2: Lie brackets for the admitted Lie point symmetries of equation (17).

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4
Y_1	0	0	Y_1	$-Y_2$
Y_2	0	0	Y_2	Y_1
Y_3	$-Y_1$	$-Y_2$	0	0
Y_4	Y_2	$-Y_1$	0	0

From above equation we have $u = w$ and $x^2 + y^2 = s$. Using these change of variables, rewrite the equation (17)

$$16s^2\frac{d^4w}{ds^4} + 64s\frac{d^3w}{ds^3} + 32\frac{d^2w}{ds^2} = \delta e^w. \tag{20}$$

Reductions of the PDE (17) for the generators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ is

$$\frac{d^4w}{ds^4} = \delta e^w. \tag{21}$$

Case III: If f is general power law, $f = \delta u^\sigma$ where $\delta = \pm 1$ and $\sigma \neq 0, 1$ then rewrite the equation (1) as

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \delta u^\sigma. \tag{22}$$

The admitted Lie point symmetry generators are given by

$$Z_1 = \frac{\partial}{\partial x}, \qquad Z_2 = \frac{\partial}{\partial y}, \qquad Z_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad Z_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{4u}{\sigma - 1}\frac{\partial}{\partial u}. \tag{23}$$

which is four dimensional Lie algebra. The commutation relations satisfied by these generators are presented in Table 3.

Table 3: Lie brackets for the admitted Lie point symmetries of equation (22).

$[Z_1, Z_j]$	Z_1	Z_2	Z_3	Z_4
Z_1	0	0	$-Z_2$	Z_1
Z_2	0	0	Z_1	Z_2
Z_3	Z_2	$-Z_1$	0	0
Z_4	$-Z_1$	$-Z_2$	0	0

Reduction under Z_1 and Z_4

In this case the two symmetry generators Z_1 and Z_4 satisfy the relation $[Z_1, Z_4] = Z_1$. This suggest that reduction in this case should start with Z_1. The similarity variables are $y = r$ and $u = w(r)$ reduce the PDE (22) to an ordinary differential equation given by

$$\frac{d^4w}{dr^4} = \delta w^\sigma. \tag{24}$$

which admits a Lagrangian $L = \frac{1}{2}w''^2 - \frac{\delta}{\sigma + 1}w^{\sigma+1}$ with variational symmetry

$$\frac{\partial}{\partial r},$$

with corresponding Noether first integral leading to

$$-w'w''' + \frac{1}{2}w''^2 + \frac{\delta}{\sigma + 1}w^{\sigma+1} = 0.$$

This ODE admits two Lie symmetries

$$Y_1 = \frac{\partial}{\partial r}, \quad Y_2 = r\frac{\partial}{\partial r} - \frac{4}{\sigma - 1} w \frac{\partial}{\partial w}.$$

and since $[Y_1, Y_2] = Y_2$, a first reduction by Y_1 leads to the second-order ODE

$$W^3 W'' + \frac{1}{2} W^2 W'^2 - \frac{\delta}{\sigma + 1} \alpha^{\sigma+1} = 0.$$

where $W = W(\alpha)$, $\alpha = w$ and $W = w'$. This second order ODE inherits the symmetry Y_2 which in the transformed variables is

$$Y_2^* = \alpha \frac{\partial}{\partial \alpha} + \frac{3 + \sigma}{4} W \frac{\partial}{\partial W}.$$

which has invariants $\beta = \frac{W}{\alpha^{\frac{3+\sigma}{4}}}$ and $V = \frac{W'}{\alpha^{\frac{\sigma-1}{4}}}$, so that the second order ODE reduces to the first order ODE, after some calculations,

$$\frac{\mathrm{d}v}{\mathrm{d}\beta} = \frac{(\sigma^2 - 1)\beta^3 v + 2(\sigma + 1)\beta^2 v^2 - 4\delta}{(\sigma + 1)\beta^3[(\sigma + 3)\beta - 4v]}.$$

The above equation can expressed in the following form

$$\frac{\mathrm{d}v}{\mathrm{d}\beta} = A_0 + \left(A_1\beta + \frac{A_2}{\beta^3}\right) v + \left(A_3 + \frac{A_4}{\beta^3}\right) v^2 + \left(A_5 + \frac{A_6}{\beta} + \frac{A_7}{\beta^3}\right) v^3. \tag{25}$$

Here $A_i, 0 \leq i \leq 7$ depend on δ and σ and are independent of s and v. If $v_0 = v_0(\beta)$ is particular solution of equation (25), the substitution $v - v_0 = \frac{1}{\phi}$ reduces it to an Abel equation [13] of second kind

$$\phi\phi'_\beta = -(3f_3 v_0^2 + 2f_2 v_0 + f_1)\phi^2 - (3f_3 v_{0+f_2})\phi - f_3, \tag{26}$$

where f_i for $0 \leq i \leq 3$ depends on β and are coefficients of v_i^3 in the equation (25). The substitution $\phi = E(\beta)\psi$ brings equation (26) to the simpler form

$$\psi\psi' = F_1(\beta)\psi + F_0(\beta), \tag{27}$$

where $F_1(\beta) = -\frac{3f_3 v_{0+f_2}}{E(\beta)}$ and $F_0(\beta) = -\frac{f_3}{E^2(\beta)}$. The equation (27) can be reduced by introducing a new independent variable $\gamma = \int F(\beta) d\beta$ to canonical form

$$\psi\psi' = \psi + \theta(\gamma), \tag{28}$$

where $\theta(\gamma) = \frac{F_0(\beta)}{F_1(\beta)}$. The solutions of equation (28) are given by [14].

The reductions of PDE (22) for Lie symmetry $\frac{\partial}{\partial y}$ is given by

$$\frac{d^4 w}{dr^4} = \delta w^\sigma, \tag{29}$$

Moreover, for the symmetry generator $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, the PDE (22) reduces to

$$16s^2 \frac{d^4 w}{ds^4} + 64s \frac{d^3 w}{ds^3} + 32 \frac{d^2 w}{ds^2} = \delta w^\sigma. \tag{30}$$

3 Noether Symmetries

The Noether symmetry or the strict variational symmetry is associated with the mechanical systems having a Lagrangian L. Moreover, the Lagrangian function L is obtained from the action integral given by

$$J[u] = \int\int_S L(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})dx\,dy. \tag{31}$$

For details about the Lagrangian and relation ship between symmetries and conservation laws see [8]. Now the Lagrangian L associated with the equation (1) is given by

$$L = g(u) - \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_{yy}^2 - u_{xy}^2. \tag{32}$$

Here, $g'(u) = f(u)$. The vector field (2) is called a Noether symmetry (variational symmetry) of Lagrangian L if the functional $\int\int_S Ldxdy$ is invariant. It turns out that, with zero gauge, X satisfies the invariance condition given by

$$X^{[2]}L + L(D_x\xi + D_y\zeta) = 0. \tag{33}$$

Here $X^{[2]}$ is the second prolongation of the vector field X given by

$$X^{[2]} = X + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_y^{(1)}\frac{\partial}{\partial u_y} + \eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}} + \eta_{xy}^{(2)}\frac{\partial}{\partial u_{xy}} + \eta_{yy}^{(2)}\frac{\partial}{\partial u_{yy}}. \tag{34}$$

As in Lie symmetry, here we also consider cases on the function $g(u)$.

Case I: If $g(u) = \delta u^\sigma$, where $\sigma \neq 0, -\frac{5}{3}$ and $\delta = \pm 1$. Then from equation (33), we get the following Noether symmetries

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad V_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad V_5 = -\frac{2}{\sigma}u\frac{\partial}{\partial u}. \tag{35}$$

Case II: If $g(u) = \delta u^{-\frac{5}{3}}$, then the admitted Noether symmetries are

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad V_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad V_5 = \frac{6}{5}u\frac{\partial}{\partial u}. \tag{36}$$

Case III: If $g(u) = \delta e^{\gamma u}$, where $\delta, \gamma = \pm 1$. In this case the Noether symmetries are

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad V_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad V_5 = \frac{2}{\gamma}\frac{\partial}{\partial u}. \tag{37}$$

Case IV: If $g(u) = \sin(u)$, then the Noether symmetries are

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad V_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad V_5 = -2\tan(u)\frac{\partial}{\partial u}. \tag{38}$$

4 Conservation Laws

Conservation laws have many significant uses in the study of differential equations. In particular, with regard to integrability and linearization, constant of motion, analysis of solutions and numerical solution methods. For variational equations, Noether's theorem can be used to construct the conserved vector (T^x, T^y). In the absence of knowledge of Noether symmetries or in non variational cases, alternative approaches may be used like the 'multiplier' method [3] and some 'homotopy' integral.

In our study we write the conservation laws as conserved vectors, i.e., if

$$D_xT^x + D_yT^y = 0, \tag{39}$$

along the solutions of the differential equation $E(x, y, u, u_x, u_y, u_{xx}, \ldots) = 0$, then (T^x, T^y) is the conserved vector and the conserved form is given by

$$T^yDx - T^xDy. \tag{40}$$

Here D_x and D_y denote the total derivative operators and are given by

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \ldots, \\ D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{yy}\frac{\partial}{\partial u_y} + \ldots. \end{aligned} \tag{41}$$

Now, we state the Noether's theorem and apply it to determine conservation laws.

Theorem 4.1. *If the vector field X in (2) is a Noether symmetry then the conserved vector (T^x, T^y) is given by*

$$T^x = L\xi + w\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} - D_y\frac{\partial L}{\partial u_{xy}}\right) + D_xw\frac{\partial L}{\partial u_{xx}} + D_yw\frac{\partial L}{\partial u_{xy}}, \tag{42}$$

$$T^y = L\zeta + w\left(\frac{\partial L}{\partial u_y} - D_x\frac{\partial L}{\partial u_{xy}} - D_y\frac{\partial L}{\partial u_{yy}}\right) + D_xw\frac{\partial L}{\partial u_{xy}} + D_yw\frac{\partial L}{\partial u_{yy}}. \tag{43}$$

Here $w = \big(\eta - u_x\xi - u_y\zeta\big)$ is the characteristic of X and D_x, D_y are the total derivative operators [3].

The conserved vectors can also be calculated by using multiplier method. For given independent variables x, y and dependent variable u, the Euler operator is defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + {D_x}^2\frac{\partial}{\partial u_{xx}} + {D_y}^2\frac{\partial}{\partial u_{yy}} + D_xD_y\frac{\partial}{\partial u_{xy}} + \ldots. \tag{44}$$

For a mth order partial differential equation

$$E(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\ldots) = 0. \tag{45}$$

The multiplier Q for (45) has the property

$$D_xT^x + D_yT^y = QE, \tag{46}$$

for arbitrary function $u(x, y)$ [12]. The determining equation for multipliers are obtained by taking the variational derivative of (46) [12]

$$\frac{\delta}{\delta u}(QE) = 0. \tag{47}$$

equation (47) holds for arbitrary function $u(x,y)$ not only for solutions of (45). We can find conserved vectors using (46) after computing the multiplier from equation (47).

Now we find the conserved vectors by using Noether theorem and by multiplier approach.

4.1 Arbitrary $g(u)$

1. $x\partial_y - y\partial_x$ - **angular momentum** :

$$\begin{aligned}&\Big(xg\left(u\right)-\frac{1}{2}xu_{xx}{}^{2}+\frac{1}{2}xu_{yy}{}^{2}-u_{x}u_{yy}-u_{yy}yu_{xy}+u_{y}u_{xy}-u_{xy}yu_{xx}-xu_{y}u_{yyy}\\&\qquad-xu_{y}u_{xxy}+yu_{x}u_{yyy}+yu_{x}u_{xxy}\Big)Dx\\&+\Big(yg\left(u\right)+\frac{1}{2}yu_{xx}{}^{2}-\frac{1}{2}yu_{yy}{}^{2}+u_{x}u_{xy}-u_{xy}xu_{yy}-u_{xx}u_{y}-u_{xx}xu_{xy}+xu_{y}u_{xyy}\\&\qquad+xu_{y}u_{xxx}-yu_{x}u_{xyy}-yu_{x}u_{xxx}\Big)Dy.\end{aligned}\tag{48}$$

2. ∂_y - **linear momentum** in y:

$$\begin{aligned}&\Big(g\left(u\right)-\frac{1}{2}u_{xx}{}^{2}+\frac{1}{2}u_{yy}{}^{2}-u_{y}u_{yyy}-u_{y}u_{xxy}\Big)Dx\\&\quad+\Big(-u_{xy}u_{yy}-u_{xy}u_{xx}+u_{y}u_{xyy}+u_{y}u_{xxx}\Big)Dy.\end{aligned}\tag{49}$$

3. ∂_x - **linear momentum** in x:

$$\begin{aligned}&\Big(u_{xy}u_{yy}+u_{xy}u_{xx}-u_{x}u_{yyy}-u_{x}u_{xxy}\Big)Dx\\&\quad+\Big(-g\left(u\right)-\frac{1}{2}u_{xx}{}^{2}+\frac{1}{2}u_{yy}{}^{2}+u_{x}u_{xyy}+u_{x}u_{xxx}\Big)Dy.\end{aligned}\tag{50}$$

4. $x\partial_x + y\partial_y$:

$$\begin{aligned}&\Big(yg\left(u\right)-\frac{1}{2}yu_{xx}{}^{2}+\frac{1}{2}yu_{yy}{}^{2}+yu_{xy}{}^{2}-2xu_{x}u_{xxy}-2yu_{y}u_{xxy}-xu_{x}u_{yyy}-yu_{y}u_{yyy}\\&\qquad+2u_{x}u_{xy}+2xu_{xx}u_{xy}+xu_{xy}u_{yy}+u_{y}u_{yy}\Big)Dx\\&+\Big(-xg\left(u\right)-\frac{1}{2}xu_{xx}{}^{2}+\frac{1}{2}xu_{yy}{}^{2}-xu_{xy}{}^{2}+xu_{x}u_{xxx}+yu_{y}u_{xxx}+2xu_{x}u_{xyy}\\&\qquad+2yu_{y}u_{xyy}-u_{x}u_{xx}-yu_{xx}u_{xy}-2u_{y}u_{xy}-2yu_{yy}u_{xy}\Big)Dy.\end{aligned}\tag{51}$$

5. $-2\dfrac{g(u)}{g'(u)}\partial_u$:

$$\begin{aligned}&\Big(-4\frac{g(u)}{g'(u)}u_{xxy} - 2\frac{g(u)}{g'(u)}u_{yyy} - 4u_x\, u_{xy} - 4\frac{g(u)g''(u)}{(g'(u))^2}u_x\, u_{xy} - 2u_y\, u_{yy} \\ &\quad - 2\frac{g(u)g''(u)}{(g'(u))^2}u_y\, u_{yy}\Big)Dx \\ &+ \Big(+2\frac{g(u)}{g'(u)}u_{xxx} + 4\frac{g(u)}{g'(u)}u_{xyy} + 2u_x\, u_{xx} + 2\frac{g(u)g''(u)}{(g'(u))^2}u_x\, u_{xx} + 4u_y\, u_{xy} \\ &\quad + 4\frac{g(u)g''(u)}{(g'(u))^2}u_y\, u_{xy}\Big)Dy.\end{aligned} \tag{52}$$

Next, we consider some special cases for function $g(u)$.

4.2 $g(u) = \delta u^\sigma$

1. $x\partial_y - y\partial_x$:

$$\begin{aligned}&\Big(x\delta\, u^\sigma - \frac{1}{2}x{u_{xx}}^2 + \frac{1}{2}x{u_{yy}}^2 - u_{yy}u_x - u_{yy}yu_{xy} + u_{xy}u_y - u_{xy}yu_{xx} - xu_yu_{yyy} \\ &\quad - xu_yu_{xxy} + yu_xu_{yyy} + yu_xu_{xxy}\Big)Dx \\ &+ \Big(y\delta\, u^\sigma + \frac{1}{2}y{u_{xx}}^2 - \frac{1}{2}y{u_{yy}}^2 + u_{xy}u_x - u_{xy}xu_{yy} - u_{xx}u_y - u_{xx}xu_{xy} + xu_yu_{xyy} \\ &\quad + xu_yu_{xxx} - yu_xu_{xyy} - yu_xu_{xxx}\Big)Dy.\end{aligned} \tag{53}$$

2. ∂_y:

$$\begin{aligned}&\Big(\delta\, u^\sigma - \frac{1}{2}{u_{xx}}^2 + \frac{1}{2}{u_{yy}}^2 - u_yu_{yyy} - u_yu_{xxy}\Big)Dx \\ &+ \Big(- u_{yy}u_{xy} - u_{xy}u_{xx} + u_yu_{xyy} + u_yu_{xxx}\Big)Dy.\end{aligned} \tag{54}$$

3. ∂_x:

$$\begin{aligned}&\Big(u_{yy}u_{xy} + u_{xy}u_{xx} - u_xu_{yyy} - u_xu_{xxy}\Big)Dx \\ &+ \Big(- \delta\, u^\sigma - \frac{1}{2}{u_{xx}}^2 + \frac{1}{2}{u_{yy}}^2 + u_xu_{xyy} + u_xu_{xxx}\Big)Dy.\end{aligned} \tag{55}$$

Additionally, for the PDEs

1.

$$\sigma u^{\sigma-1} - u_{xxxx} - u_{yyyy} - 2u_{xxyy} = 0, \tag{56}$$

we have additional conservation laws, in fact, infinitely many if we consider higher-order cases.

Firstly, with additional first-order multipliers, we get, for e.g.,

a. $Q = e^x$:

$$\begin{aligned} &-\frac{1}{3}\mathrm{e}^x\Big(3\,u_{yyy} + 3\,u_{xxy} - 2\,u_{xy} + u_y\Big)Dx \\ &\quad -\frac{1}{3}\mathrm{e}^x\Big(-3\,u_{xyy} - 3\,u_{xxx} + u_{yy} + 3\,u_{xx} - 3\,u_x + 3\,u\Big)Dy. \end{aligned} \tag{57}$$

b. $Q = \sin x$:

$$\begin{aligned} &\Big(-u_{yyy}\sin(x) - u_{xxy}\sin(x) + 2/3\,u_{xy}\cos(x) + \frac{1}{3}u_y\sin(x)\Big)Dx \\ &+\Big(u_{xyy}\sin(x) + u_{xxx}\sin(x) - \frac{1}{3}u_{yy}\cos(x) - u_{xx}\cos(x) - u_x\sin(x) + u\cos(x)\Big)Dy, \end{aligned} \tag{58}$$

and an example of a higher-order multiplier/conservation law is

c. $Q = u_{xxy}$:

$$\begin{aligned} &\Big(-\frac{1}{2}u_{yyy}u_{xxy} - \frac{1}{2}{u_{xxy}}^2 + \frac{1}{2}u_{yy}u_{xxyy} + \frac{1}{3}u_{xy}u_{xxxy} - \frac{1}{6}u_{xx}u_{xxyy} + \frac{1}{3}u_{xx}u \\ &\quad -\frac{1}{6}u_{xx}u_{xxxx} - \frac{1}{6}u_{xx}u_{yyyy} - \frac{1}{2}u_yu_{xxyyy} - \frac{1}{6}u_yu_{xxxxy} - \frac{1}{6}{u_x}^2 + \frac{1}{6}u_xu_{xxxxx} \\ &\quad +\frac{1}{6}u_xu_{xyyyy} + \frac{1}{3}uu_{xxyyyy} + \frac{1}{6}uu_{xxxxyy} - \frac{1}{6}uu_{xxxxxx}\Big)Dx \\ &\Big(\frac{1}{2}u_{xyy}u_{xxy} + \frac{1}{2}u_{xxx}u_{xxy} - \frac{1}{6}u_{yy}u_{xxxy} + \frac{1}{3}u_{xy}u_{xxyy} - \frac{2}{3}\,u_{xy}u + \frac{1}{3}u_{xy}u_{xxxx} \\ &\quad +\frac{1}{3}u_{xy}u_{yyyy} - \frac{1}{2}u_{xx}u_{xxxy} + \frac{1}{3}u_yu_x - \frac{1}{6}u_yu_{xxxxx} - \frac{1}{6}u_yu_{xyyyy} - \frac{1}{6}u_xu_{xxyyy} \\ &\quad +\frac{1}{3}u_xu_{xxxxy} - \frac{1}{6}u_xu_{yyyyy} + \frac{1}{6}uu_{xxxyyy} - \frac{1}{6}uu_{xxxxxy} + \frac{1}{3}uu_{xyyyyy}\Big)Dy. \end{aligned} \tag{59}$$

2.

$$\sigma\delta u^{\sigma-1} - u_{xxxx} - u_{yyyy} - 2u_{xxyy} = 0, \qquad \sigma = -2, \tag{60}$$

(a) $Q = x^2u_y - y^2u_y + 2\,yu - 2\,yxu_x$:

$$\begin{aligned}
-\frac{1}{12u^2}\Bigg[\Big(&8\,u_y{}^2u^2 + 4\,u_{xx}u^3 - 12\,u_{yy}u^3 + 3\,x^2u_{xxyy}u^3 - 3\,y^2u_{xxyy}u^3 \\
&+ 3\,x^2u_{xxxx}u^3 - 3\,y^2u_{xxxx}u^3 - 2\,u_{xy}{}^2u^2x^2 - 10\,u_{xxy}u^2yxu_x \\
&- 6\,u_{yyy}u^2yxu_x + 6\,u_{xy}u^2yxu_{xx} + 6\,u_{yy}u^2yxu_{xy} - 6\,u_yu^2yxu_{xyy} \\
&- 2\,u_yu^2yxu_{xxx} + 6\,yxu_{xxxy}u^3 + 6\,yxu_{xyyy}u^3 - 6\,u_yu^2yu_{yy} \\
&- 2\,u_yu^2yu_{xx} + 4\,u_{xxy}u^2x^2u_y - 4\,u_{xxy}u^2y^2u_y + 10\,u_{yy}u^2xu_x \\
&- 12\,u_{xy}u^2xu_y - u_{xx}u^2u_{yy}x^2 + u_{xx}u^2u_{yy}y^2 - 2\,u_{xx}u^2xu_x \\
&- 4\,u_xu^2u_{xy}y + 6\,u_{yyy}u^2x^2u_y - 6\,u_{yyy}u^2y^2u_y - 2\,u_xu^2u_{xyy}y^2 \\
&+ 2\,u_xu^2u_{xyy}x^2 + 3\,u_{yy}{}^2u^2y^2 + 18\,u_{yyy}yu^3 + 6\,xu_{xxx}u^3 + 2\,u_{xy}{}^2u^2y^2 \\
&- 3\,u_{yy}{}^2u^2x^2 + 6\,u_{xyy}xu^3 + 18\,u_{xxy}yu^3 + 6\,x^2\delta - 6\,y^2\delta\Big)Dx \\
\Big(&3\,u_{xyyy}x^2u^3 - 3\,u_{xyyy}y^2u^3 + 6\,u_{yyy}xu^3 + 3\,u_{xxxy}x^2u^3 - 3\,u_{xxxy}y^2u^3 \\
&+ 6\,u_{xxy}xu^3 - 8\,u_xu^2u_y - 18\,yu_{xxx}u^3 - 18\,u_{xyy}yu^3 + 8\,u_{xyy}u^2yxu_x \\
&- 2\,u_{yy}u^2yxu_{xx} + 4\,u_yu^2u_{xxy}yx + 12\,u_{xxx}u^2yxu_x + 6\,yxu^3u_{yyyy} \\
&+ 6\,yxu^3u_{xxyy} + 3\,u_xu^2u_{xxy}y^2 + 5\,u_{xyy}u^2y^2u_y - 5\,u_{xyy}u^2x^2u_y \\
&- 3\,u_{xxx}u^2x^2u_y + 3\,u_{xxx}u^2y^2u_y - 2\,u_{yy}u^2xu_y + 3\,u_{yy}u^2u_{xy}x^2 \\
&- 3\,u_{yy}u^2u_{xy}y^2 - 12\,u_{xy}u^2xu_x - 4\,u_{xy}{}^2u^2yx + 10\,u_{xx}u^2xu_y \\
&- 6\,u_{xx}{}^2u^2yx + 3\,u_{xx}u^2u_{xy}x^2 - 3\,u_{xx}u^2u_{xy}y^2 + 4\,u_yu^2u_{xy}y \\
&+ 2\,u_xu^2yu_{yy} + u_xu^2u_{yyy}y^2 - u_xu^2u_{yyy}x^2 + 6\,u_xu^2yu_{xx} \\
&- 3\,u_xu^2u_{xxy}x^2 + 16\,u_{xy}u^3 + 12\,yx\delta\Big)Dy\Bigg].
\end{aligned} \tag{61}$$

4.2.1 $g(u) = \delta u^{-\frac{5}{3}}$

i. $x\partial_y - y\partial_x$:

$$\begin{aligned}
-\frac{1}{2u^{\frac{5}{3}}}\Big(&-2\,x\delta + xu_{xx}{}^2u^{5/3} - xu_{yy}{}^2u^{5/3} + 2\,u_xu_{yy}u^{5/3} + 2\,u_{yy}yu_{xy}u^{5/3} - 2\,u_yu_{xy}u^{5/3} \\
&+ 2\,u_{xy}yu_{xx}u^{5/3} + 2\,xu_yu_{yyy}u^{5/3} + 2\,xu_yu_{xxy}u^{5/3} - 2\,yu_xu_{yyy}u^{5/3} \\
&- 2\,yu_xu_{xxy}u^{5/3}\Big)Dx \\
+\frac{1}{2u^{\frac{5}{3}}}\Big(&2\,y\delta + yu_{xx}{}^2u^{5/3} - yu_{yy}{}^2u^{5/3} + 2\,u_xu_{xy}u^{5/3} - 2\,u_{xy}xu_{yy}u^{5/3} - 2\,u_{xx}u_yu^{5/3} \\
&- 2\,u_{xx}xu_{xy}u^{5/3} + 2\,xu_yu_{xyy}u^{5/3} + 2\,xu_yu_{xxx}u^{5/3} - 2\,yu_xu_{xyy}u^{5/3} \\
&- 2\,yu_xu_{xxx}u^{5/3}\Big)Dy.
\end{aligned} \tag{62}$$

ii. ∂_y:

$$-\frac{1}{2u^{\frac{5}{3}}}\Big(-2\,\delta+{u_{xx}}^2u^{5/3}-{u_{yy}}^2u^{5/3}+2\,u_yu_{yyy}u^{5/3}+2\,u_yu_{xxy}u^{5/3}\Big)Dx \\ \Big(-u_{xy}u_{yy}-u_{xy}u_{xx}+u_yu_{xyy}+u_yu_{xxx}\Big)Dy. \quad (63)$$

iii. ∂_x:

$$\Big(u_{xy}u_{yy}+u_{xy}u_{xx}-u_xu_{yyy}-u_xu_{xxy}\Big)Dx \\ \frac{1}{2u^{\frac{5}{3}}}\Big(-2\,\delta-{u_{xx}}^2u^{5/3}+{u_{yy}}^2u^{5/3}+2\,u_xu_{xyy}u^{5/3}+2\,u_xu_{xxx}u^{5/3}\Big)Dy. \quad (64)$$

4.3 $g(u)=\sin(u)$

i. $x\partial_y-y\partial_x$:

$$\Big(x\sin(u)-\frac{1}{2}x{u_{xx}}^2+\frac{1}{2}x{u_{yy}}^2-u_{yy}u_x-u_{yy}yu_{xy}+u_{xy}u_y-u_{xy}yu_{xx}-xu_yu_{yyy} \\ -xu_yu_{xxy}+yu_xu_{yyy}+yu_xu_{xxy}\Big)Dx \\ +\Big(y\sin(u)+\frac{1}{2}\,y{u_{xx}}^2-\frac{1}{2}y{u_{yy}}^2+u_{xy}u_x-u_{xy}xu_{yy}-u_{xx}u_y-u_{xx}xu_{xy}+xu_yu_{xyy} \\ +xu_yu_{xxx}-yu_xu_{xyy}-yu_xu_{xxx}\Big)Dy. \quad (65)$$

ii. ∂_y:

$$\Big(\sin(u)-\frac{1}{2}{u_{xx}}^2+\frac{1}{2}{u_{yy}}^2-u_yu_{yyy}-u_yu_{xxy}\Big)Dx \\ +\Big(-u_{yy}u_{xy}-u_{xy}u_{xx}+u_yu_{xyy}+u_yu_{xxx}\Big)Dy. \quad (66)$$

iii. ∂_x:

$$\Big(u_{xy}u_{yy}+u_{xy}u_{xx}-u_xu_{yyy}-u_xu_{xxy}\Big)Dx \\ +\Big(-\sin(u)-\frac{1}{2}{u_{xx}}^2+\frac{1}{2}{u_{yy}}^2+u_xu_{xyy}+u_xu_{xxx}\Big)Dy. \quad (67)$$

5 Conclusions

In this paper, we have classified the symmetries and conservation laws of the fourth-order biharmonic PDE; the technique presented here set the scene for further interesting studies of high-order nonlinear PDEs of mathematical physics. One can consider time dependent biharmonic equation in $(2+1)$ dimension.

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References

[1] A. Ahmad, A. H. Bokhari, A. Kara & F. Zaman (2008). Symmetry classifications and reductions of some classes of (2+ 1)-nonlinear heat equation. *Journal of Mathematical Analysis and Applications*, *339*(1), 175–181.

[2] S. Anco & G. Bluman (2002). Examples of conservation law classifications. *European Journal of Applied Mathematics*, *9*, 567–585.

[3] G. Bluman & S. Anco (2008). *Symmetry and integration methods for differential equations* volume 154. Springer Science & Business Media, New York.

[4] A. H. Bokhari, F. Mahomed & F. Zaman (2010). Symmetries and integrability of a fourth-order Euler–Bernoulli beam equation. *Journal of Mathematical Physics*, *51*(5), 053517. https://doi.org/10.1063/1.3377045.

[5] N. H. Ibragimov, A. H. Kara & F. M. Mahomed (1998). Lie–Bäcklund and Noether symmetries with applications. *Nonlinear Dynamics*, *15*, 115–136. https://doi.org/10.1023/A:1008240112483.

[6] N. H. Ibragimov & T. Kolsrud (2004). Lagrangian approach to evolution equations: symmetries and conservation laws. *Nonlinear Dynamics*, *36*, 29–40. https://doi.org/10.1023/B:NODY.0000034644.82259.1f.

[7] A. Johnpillai, C. Khalique & F. Mahomed (2019). Travelling wave group-invariant solutions and conservation laws for $theta$-equation. *Malaysian Journal of Mathematical Sciences*, *13*(1), 13–29.

[8] A. H. Kara & F. M. Mahomed (2000). Relationship between symmetries and conservation laws. *International Journal of Theoretical Physics*, *39*, 23–40. https://doi.org/10.1023/A:1003686831523.

[9] A. H. Kara & F. M. Mahomed (2002). A basis of conservation laws for partial differential equations. *Journal of Nonlinear Mathematical Physics*, *9*(sup2), 60–72. https://doi.org/10.2991/jnmp.2002.9.s2.6.

[10] P. G. L. Leach & A. Paliathanasis (2021). Symmetry analysis for a fourth-order noise-reduction partial differential equation. *Quaestiones Mathematicae*, *44*(11), 1541–1552. https://doi.org/10.2989/16073606.2020.1812009.

[11] A. E. H. Love (1927). *Treatise on the mathematical theory of elasticity*. University Press, Cambridge, UK.

[12] P. J. Olver (1993). *Applications of Lie groups to differential equations* volume 107. Springer Science & Business Media, New York, NY.

[13] D. E. Panayotoukanos, N. D. Panayotounakou & A. F. Vakakis (2002). On the solution of the unforced damped Duffing oscillator with no linear stiffness term. *Nonlinear Dynamics*, *28*, 1–16. https://doi.org/10.1023/A:1014925032022.

[14] A. D. Polyanin & V. F. Zaitsev (2002). *Handbook of exact solutions of ordinary differential equations*. Chapman and Hall, London.

[15] A. Selvadurai & A. Selvadurai (2000). The biharmonic equation. *Partial Differential Equations in Mechanics 2: The Biharmonic Equation, Poisson's Equation*, pp. 1–502.

[16] N. Spirana & W. Chatanin (2016). Lie symmetry analysis and exact solutions to the Quintin nonlinear Beam equation. *Malaysian Journal of Mathematical Sciences*, *10*(1), 61–68.

[17] T. Wolf (2002). A comparison of four approaches to the calculation of conservation laws. *European Journal of Applied Mathematics*, *13*(2), 129–152. https://doi.org/10.1017/S0956792501004715.